Euler Estimates for Rough Differential Equations

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Abstract

We consider controlled differential equations and give new estimates for higher order Euler schemes. Our proofs are inspired by recent work of A. M. Davie who considers first and second order schemes. In order to implement the general case we make systematic use of geodesic approximations in the free nilpotent group.

As application, we can control moments of solutions to rough path differential equations (RDEs) driven by random rough paths with sufficient integrability and have a criteria for L^q-convergence in the Universal Limit Theorem. We also obtain Azencott type estimates and asymptotic expansions for random RDE solution. When specialized to RDEs driven by Enhanced Brownian motion, we (mildly) improve classic estimates for diffusions in the small time limit.

Introduction

We consider controlled differential equations of the form

$$dy = V_1(y)\,dx^1 + ... + V_d(y)\,dx^d$$

on the time interval $[0,1]$. When the $\mathbb{R}^d$-valued driving signal x and the vector fields are Lipschitz continuous then there exists a unique solution for every starting point y_0. Building on recent work by A. M. Davie [7], we control the Hölder norm of y in terms of suited Hölder norms of x and its iterated integrals and also obtain higher order Euler estimates. To be able to go beyond the first and second order case discussed by Davie we use ideas from sub-Riemannian geometry. En passant, we construct *geodesic approximations* which respect the geometry of the vector fields. Finally, by passing to the limit, all estimates extend to solutions of rough path differential equations (RDEs) in the sense of T. Lyons.

As application, we show that random RDE solutions driven by a sufficiently integrable geometric rough path are in $L^q(\Omega)$ for all $q<\infty$. The examples we have in mind are RDEs driven by *Enhanced Brownian motion* (in which case we are effectively dealing with a Stratonovich stochastic differential equation) as well as certain *Enhanced Gaussian* - and *Enhanced Markov processes.*

When specialized to RDEs driven by Enhanced Brownian motion, the above mentioned Euler estimates are closely related to classic estimates by Azencott. In fact, at least in absence of a drift vector field, our main Euler estimate sharpens a key result in [1] when applied to *diffusions en temps petit,* which is precisely the result which is quoted and used in Ben Arous' and, later, Castell's work [2, 4].

As the reader may suspect, the robustness of the rough path approach allows to obtain Azencott type estimates for *arbitrary* random RDE solutions driven by a sufficiently integrable geometric rough path. The case of RDEs driven by EBM, as discussed in the last paragraph, is just a special case of this general result.

Notation 1 *The dimensions of $\mathbb{R}^d, \mathbb{R}^e$ are fixed and will not appear explicitly when we write out the dependence of constants. In general, constants that appear in lemmas, propositions, theorems etc have an index that matches the number of the statement. In the proofs we indicate changing constants by a running upper index.*

1 Preliminaries: Euler scheme of order N

Define $T^{(N)} = \oplus_{k=0}^{N} \left(\mathbb{R}^d\right)^{\otimes k}$, by convention $\left(\mathbb{R}^d\right)^{\otimes 0} \equiv \mathbb{R}$. Let x be an $\mathbb{R}^d$-valued Lipschitz path and define the k^{th} iterated integrals of the path segment $x|_{[s,t]}$ as

$$\mathbf{g}^{k,i_1,\cdots,i_k} := \int_s^t \int_s^{u_k} \cdots \int_s^{u_2} dx_{u_1}^{i_1} \ldots dx_{u_k}^{ik}.$$

and so that $\mathbf{g}^k = \left(\mathbf{g}^{k,i_1,\cdots,i_k}\right)_{i_1,\cdots,i_k \in \{1,\ldots,d\}} \in \left(\mathbb{R}^d\right)^{\otimes k}$. For later convenience set $\mathbf{g}^0 = 1 \in \left(\mathbb{R}^d\right)^{\otimes 0} \equiv \mathbb{R}$. We then define the (step-N) signature of the path segment $x|_{[s,t]}$ as

$$\mathbf{x}_{s,t} \equiv S_N(x)_{s,t} \equiv 1 + \sum_{k=1}^{N} \mathbf{g}^k \in T^{(N)}\left(\mathbb{R}^d\right).$$

We say that a vector field is in $\mathrm{Lip}^\gamma(\mathbb{R}^e)$ if it has $\lfloor \gamma \rfloor$ bounded derivatives and the $\lfloor \gamma \rfloor^{th}$-derivative is $\{\gamma\}$-Hölder continuous.

Definition 2 *Given vector fields $V_1, \ldots, V_d \in \mathrm{Lip}^1(\mathbb{R}^e)$ (= bounded & Lipschitz continuous vector fields) and a $\mathbb{R}^d$-valued Lipschitz path x on $[0,1]$, we let $y = \pi(0, y_0; x)$ denote the unique solution to the (control) ODE*

$$dy_t = \sum_{i=1}^{d} V_i(y_t)\, dx_t^i \equiv V(y_t)\, dx_t, \ \ t \in [0,1].$$

started at y_0.

The following lemma is left as a simple exercise.

Lemma 3 *Assume that* $(V_i)_{1\leq i\leq d} \in \mathrm{Lip}^1(\mathbb{R}^e)$. *Let* x *be an* $\mathbb{R}^d$*-valued Lipschitz path on* $[0,1]$ *and let* $y_t = \pi(0, y_0; x)_t$. *Then, for all* $0 \leq s \leq t \leq 1$,

$$|y_{s,t}| \leq C_3 \int_s^t |dx_r|$$

where C_3 *depends on (the Lipschitz norm of) the vector fields* $V_1, ...V_d$.

Let us now define the Euler approximation of order N to a control ODE of the above type. To this end, let H denote the identity function on $\mathbb{R}^e$ and recall the identification of vector fields with first order differential operators.

Definition 4 *Given* $(V_i)_{1\leq i\leq d} \in \mathrm{Lip}^N(\mathbb{R}^e)$, $\mathbf{g} \in T^{(N)}(\mathbb{R}^d)$ *and* $y \in \mathbb{R}^e$ *we call*

$$I^{y,N,\mathbf{g}} := I[y,N,\mathbf{g}] := \sum_{k=1}^{N} \sum_{\substack{i_1,\dots,i_k \\ \in\{1,\dots,d\}}} V_{i_1}\cdots V_{i_k} H(y)\, \mathbf{g}^{k,i_1,\cdots,i_k},$$

the (increment of) the step-N Euler scheme.

This definition is explained by

Lemma 5 *Assume that* $(V_i)_{1\leq i\leq d} \in \mathrm{Lip}^N(\mathbb{R}^e)$. *Let* x *be an* $\mathbb{R}^d$*-valued Lipschitz path on* $[0,1]$ *and let* $y_t = \pi(0, y_0; x)_t$. *Then, for all* $0 \leq s \leq t \leq 1$,

$$\begin{aligned} & y_{s,t} - I^{y_s,N,S_N(x)_{s,t}} \\ = & \sum_{\substack{i_1,\dots,i_N \\ \in\{1,\dots,d\}}} \int_{s<r_1<\ldots<r_N<t} \left[V_{i_1}\cdots V_{i_N} H(y_{r_1}) - V_{i_1}\cdots V_{i_N} H(y_s)\right] dx_{r_1}^{i_1}\cdots dx_{r_N}^{i_N} \end{aligned}$$

and there exists a constant C_5 *depending on* N *and* $V_1, ..., V_d$ *such that*

$$\left|y_{s,t} - I^{y_s,N,\mathbf{x}_{s,t}}\right| \leq C_5 \left(\int_s^t |dx_r|\right)^{N+1}.$$

Proof. Let f be smooth and note that $V_i \in \mathrm{Lip}^N$ implies $V_{i_1}\cdots V_{i_k} f$ is C^1 for $1 \leq k \leq N$. Iterated use of the fundamental theorem of calculus gives

$$\begin{aligned} f(y_t) &= f(y_s) + \sum_{k=1}^{N-1} \sum_{\substack{i_1,\dots,i_k \\ \in\{1,\dots,d\}}} \int_{s<r_1<\ldots<r_k<t} V_{i_1}\cdots V_{i_k} f(y_s)\, dx_{r_1}^{i_1}\cdots dx_{r_k}^{i_k} \\ &\quad + \sum_{\substack{i_1,\dots,i_k \\ \in\{1,\dots,d\}}} \int_{s<r_1<\ldots<r_N<t} V_{i_1}\cdots V_{i_N} f(y_{r_1})\, dx_{r_1}^{i_1}\cdots dx_{r_N}^{i_N}. \end{aligned}$$

This first part is then proved by specializing to $f = H$. For the second statement, lemma 3 gives

$$|y_{s,t}| \leq C_5^1 \int_s^t |dx_r| \,.$$

Lip^N-regularity of the vector fields implies that $V_{i_1}..V_{i_N} H(\cdot)$ is Lipschitz and hence, for $r \in [s,t]$,

$$|V_{i_1}..V_{i_N} H(y_r) - V_{i_1}..V_{i_N} H(y_s)| \leq C_5^2 \int_s^t |dx_r| \,.$$

This leads to

$$\left| \int_{s<r_1<\ldots<r_N<t} [V_{i_1}..V_{i_N} H(y_{r_1}) - V_{i_1}..V_{i_N} H(y_s)]\, dx_{r_1}^{i_1} \cdots dx_{r_N}^{i_N} \right| \leq C_5^3 \left(\int_s^t |dx_r| \right)^{N+1}$$

and summing over the indices finishes the proof. ■

2 Preliminaries II: Algebra of Iterated Integrals

The set $T_1^N(\mathbb{R}^d) \equiv \left\{ \mathbf{g} \in T^N(\mathbb{R}^d) : \mathbf{g}^0 = 1 \right\}$ is a group under *truncated tensor multiplication:* if $\mathbf{g} = 1 + \mathbf{g}^1 + \ldots + \mathbf{g}^N \equiv 1 + \tilde{\mathbf{g}}$ and similar for $\mathbf{h}$ then for $k = 0, \ldots, N$

$$(\mathbf{g} \otimes \mathbf{h})^k = \sum_{i=0}^k \mathbf{g}^i \otimes \mathbf{h}^{k-i}.$$

The neutral element is $\mathbf{e} = 1 = 1 + 0 + \ldots + 0$ and the inverse is given by the usual power series calculus

$$(1 + \tilde{\mathbf{g}})^{-1} = 1 - \tilde{\mathbf{g}} + \tilde{\mathbf{g}}^{\otimes 2} - \ldots$$

For every $\lambda \in \mathbb{R}$, the *dilatation* map δ_λ is defined componenwise by $\mathbf{g}^k \mapsto \lambda^k \mathbf{g}^k$, $k = 0, \ldots, N$.

$$\delta_\lambda : \left(\mathbf{g}^k\right) \mapsto \left(\lambda^k \mathbf{g}^k\right), \qquad \lambda \in \mathbb{R}.$$

Obviously, $T_1^N(\mathbb{R}^d)$ is a Lie group. Its Lie algebra can be identified with

$$T_0^N(\mathbb{R}^d) \equiv \left\{ \tilde{\mathbf{g}} \in T^N(\mathbb{R}^d) : \tilde{\mathbf{g}}^0 = 0 \right\}, \ \left[\tilde{\mathbf{g}}, \tilde{\mathbf{h}}\right] = \tilde{\mathbf{g}} \otimes \tilde{\mathbf{h}} - \tilde{\mathbf{h}} \otimes \tilde{\mathbf{g}}$$

and the exponential map with $\exp : T_0^N(\mathbb{R}^d) \to T_1^N(\mathbb{R}^d)$, $\tilde{\mathbf{g}} \mapsto 1 + \tilde{\mathbf{g}} + \frac{1}{2!}\tilde{\mathbf{g}}^{\otimes 2} + \ldots$

We recall some well-known facts. See [15, 10, 18, 3] for further references.

Proposition 6 (Chen, [15]) *Let $x : [0,1] \to \mathbb{R}^d$ be Lipschitz continuous with (step-N) signatures $\mathbf{x}_{s,t} = S_N(x)_{s,t}$. Then*

$$S_N(x)_{s,t} \otimes S_N(x)_{t,u} = S_N(x)_{s,u} \,. \tag{1}$$

We define $G^N(\mathbb{R}^d) \equiv \exp(L^N(\mathbb{R}^d))$ where

$$L = L^N(\mathbb{R}^d) = \mathbb{R}^d \oplus [\mathbb{R}^d, \mathbb{R}^d] \oplus [\mathbb{R}^d, [\mathbb{R}^d, \mathbb{R}^d]] \oplus ... \subset T_0^N(\mathbb{R}^d),$$

$G^N(\mathbb{R}^d)$ is a Lie subgroup of $T_1^N(\mathbb{R}^d)$ with respect to $\otimes$-multiplication and known as *step-N nilpotent free group over $\mathbb{R}^d$*.

Theorem 7 (Chow, [18]) *For every $\mathbf{g} \in G^N(\mathbb{R}^d)$ there exists an $\mathbb{R}^d$-valued Lipschitz path x such that $S_N(x)_{0,1} = \mathbf{g}$. More precisely, G is the group generated by $\{\exp(v) : v \in \mathbb{R}^d\}$ so that every $\mathbf{g} \in G$ is the signature of a (finite number of) concatenation of straight path segments.*

Theorem 8 (Geodesic Existence, [18]) *For every $\mathbf{g} \in G^N(\mathbb{R}^d)$,*

$$\|\mathbf{g}\| := \inf\left\{ \int_0^1 |\dot{\gamma}_t|\, dt : \gamma : [0,1] \to \mathbb{R}^d \text{ Lipschitz continuous, } \gamma(0) = 0,\ S_N(\gamma)_{0,1} = \mathbf{g} \right\}$$

is finite and achieved at some minimizing Lipschitz continuous path γ^*, i.e.

$$\|\mathbf{g}\| = \int_0^1 |\dot{\gamma}_t^*|\, dt \text{ and } S_N(\gamma^*)_{0,1} = \mathbf{g}.$$

Moreover, by simple reparametrization, we can state that for every $s, t \in \mathbb{R}$ with $s < t$ then exists a Lipschitz path $x^{s,t} : [s,t] \to \mathbb{R}^d$ with signature $\mathbf{g}$ and length $\|\mathbf{g}\|$:

$$S_N(x^{s,t})_{s,t} = \mathbf{g} \quad \text{and} \quad \int_s^t |dx^{s,t}| = \|\mathbf{g}\|.$$

Remark 9 *$G^N(\mathbb{R}^d)$ can be given a subriemannian structure so that the path $t \in [0,1] \mapsto S_N(\gamma^*)_{0,t}$ is a subriemannian geodesic which connects the unit e with $\mathbf{g} \in G^N(\mathbb{R}^d)$, see [3, 18]. Thus, strictly speaking, γ^* is not a geodesics but the projection of a geodesic.*

The geodesic existence theorem has useful consequences. If $\mathbf{g}, \mathbf{h} \in G^N(\mathbb{R}^d)$ then (i) $\|\mathbf{g}\| = 0$ iff $\mathbf{g} = e$, (ii) symmetry: $\|\mathbf{g}\| = \|\mathbf{g}^{-1}\|$, (iii) sub-additivity $\|\mathbf{g} \otimes \mathbf{h}\| \leq \|\mathbf{g}\| + \|\mathbf{h}\|$ and (iv) homogenity $\|\delta_\lambda \mathbf{g}\| = |\lambda|\, \|\mathbf{g}\|$ for all $\lambda \in \mathbb{R}$, hold true. In particular, $d(\mathbf{g}, \mathbf{h}) := \|\mathbf{g}^{-1} \otimes \mathbf{h}\|$ defines a left-invariant metric on $G^N(\mathbb{R}^d)$, the *Carnot-Caratheodory metric.*

Theorem 10 ([18]) *(a) The topology induced by Carnot-Caratheodory metric coincides with the manifold topology of $G^N(\mathbb{R}^d)$ and the trace topology as as subset of $T_1^N(\mathbb{R}^d)$.*
(b) The map $\mathbf{g} \mapsto \|\mathbf{g}\|$ is continuous in this topology.
(c) The space $G^N(\mathbb{R}^d)$ with metric d is Polish.

Proposition 11 ([11]) *Let $|||.|||_i$ $(i = 1, 2)$ be continuous homogenous norms on $G^N(\mathbb{R}^d)$, that is, norms that satisfies properties (i) and (iv) and such that*

$\mathbf{g} \mapsto |||\mathbf{g}|||_i$ is continuous w.r.t. τ. Then there exists a constant $c \in [1, \infty)$ such that $|||.|||_1 \sim |||.|||_2$ by which we mean

$$\frac{1}{c} |||.|||_2 \leq |||.|||_1 \leq c\, |||.|||_2 .$$

For instance,

$$|||\mathbf{g}||| \equiv \max_{k=1,\ldots,N} \left| \mathbf{g}^k \right|^{1/k} .$$

provides a useful example of a continuous homogenous norm on $G^N\left(\mathbb{R}^d\right)$ other than $\|\cdot\|$.

3 Preliminaries III: Geometric (Hölder) Rough Paths

Here, and in the remainder of this paper, we work exclusively with Hölder modulus

$$\omega(s,t) \equiv t - s.$$

Let $p \in [1, \infty)$. A path $\mathbf{x}$ from $[0,1]$ to $G^N\left(\mathbb{R}^d\right)$ is $1/p$-Hölder continuous if for all $s, t \in [0,1]$

$$\|\mathbf{x}_{s,t}\| \leq C \omega(s,t)^{1/p}$$

for some constant C. This class is denoted by $C^{1/p\text{-Hölder}}\left([0,1], G^N\left(\mathbb{R}^d\right)\right)$. We can restrict attention to paths with pinned starting point. The (homogenous) $1/p$-Hölder "norm" (there is no linear space here) on $C^{1/p\text{-Hölder}}\left([0,1], G^N\left(\mathbb{R}^d\right)\right)$ is defined by

$$\|\mathbf{x}\|_{1/p\text{-Hölder};[0,1]} = \|\mathbf{x}\|_{1/p\text{-Hölder}} = \sup_{0 \leq s < t \leq 1} \frac{\|\mathbf{x}_{s,t}\|}{\omega(s,t)^{1/p}}$$

and there is a $1/p$-Hölder metric based on the CC-metric,

$$d_{1/p\text{-Hölder};[0,1]}(\mathbf{x}, \tilde{\mathbf{x}}) = d_{1/p\text{-Hölder}}(\mathbf{x}, \tilde{\mathbf{x}}) = \sup_{0 \leq s < t \leq 1} \frac{d\left(\mathbf{x}_{s,t}, \tilde{\mathbf{x}}_{s,t}\right)}{\omega(s,t)^{1/p}}.$$

We also set

$$d_{\infty;[0,1]}(\mathbf{x}, \tilde{\mathbf{x}}) = d_{\infty}(\mathbf{x}, \tilde{\mathbf{x}}) = \sup_{0 \leq s < t \leq 1} d\left(\mathbf{x}_{s,t}, \tilde{\mathbf{x}}_{s,t}\right).$$

Theorem 12 ([10]) *(i) $C^{1/p\text{-Hölder}}\left([0,1], G^N\left(\mathbb{R}^d\right)\right)$ is a complete metric space under the metric $d_{1/p\text{-Hölder}}$.*
(ii) Every $1/p$-Hölder continuous path $\mathbf{x} \in C^{1/p\text{-Hölder}}\left([0,1], G^N\left(\mathbb{R}^d\right)\right)$ can be approximated by Lipschitz paths $x_n : [0,1] \to \mathbb{R}^d$ in the sense that

$$S_N(x_n) \to \mathbf{x} \quad \textit{uniformly on } [0,1]$$

and $\sup_n \|S_N(x_n)\|_{1/p\text{-Hölder}} < \infty$. *In fact, we can find Lipschitz paths* x_n *such that*

$$\sup_n \|S_N(x_n)\|_{1/p\text{-Hölder}} \leq 3\|\mathbf{x}\|_{1/p\text{-Hölder}}.$$

(iii) Assume $p > 1$. *Define* $C^{0,1/p\text{-Hölder}}([0,1], G^N(\mathbb{R}^d))$ *as the closure of lifted Lipschitz paths* $S_N(x)$ *under the metric* $d_{1/p\text{-Hölder}}$. *For* $\mathbf{x} \in C^{1/p\text{-Hölder}}([0,1], G^N(\mathbb{R}^d))$ *we have*

$$\mathbf{x} \in C^{0,1/p\text{-Hölder}}([0,1], G^N(\mathbb{R}^d)) \text{ iff } r(\delta;\mathbf{x}) \equiv \sup_{\substack{0\leq s<t\leq 1\\ t-s\leq\delta}} \frac{\|\mathbf{x}_{s,t}\|}{\omega(s,t)^{1/p}} \to 0 \text{ as } \delta \to 0.$$

In particular, $C^{0,1/p\text{-Hölder}}([0,1], G^N(\mathbb{R}^d)) \subsetneq C^{1/p\text{-Hölder}}([0,1], G^N(\mathbb{R}^d))$.

One can see that $C^{0,1/p\text{-Hölder}}([0,1], G^N(\mathbb{R}^d))$ is Polish whereas $C^{1/p\text{-Hölder}}([0,1], G^N(\mathbb{R}^d))$ lacks separability. Recall that $[p]$ denotes the integer part of some (positive) real number p.

Definition 13 ([15, 10]) *A path in* $C^{1/p\text{-Hölder}}([0,1], G^{[p]}(\mathbb{R}^d))$ *is called a weak geometric* p*-rough path (with Hölder-control* ω*). A path in* $C^{0,1/p\text{-Hölder}}([0,1], G^{[p]}(\mathbb{R}^d))$ *is called a geometric* p*-rough path (with Hölder- control* ω*).*

Proposition 14 ([15, 17]) *Let* $\mathbf{x} \in C^{1/p\text{-Hölder}}([0,1], G^{[p]}(\mathbb{R}^d))$ *and* $N > [p]$*.Then there exists a unique lift of* $\mathbf{x}$ *to a* $G^N(\mathbb{R}^d)$*-valued* $1/p$*-Hölder continuous path w.r.t. Carnot-Caratheodory metric on* $G^N(\mathbb{R}^d)$*, denoted by* $S_N(\mathbf{x}) : [0,1] \to G^N(\mathbb{R}^d)$*. Moreover, there exists a const* $C_{14} = C(p,N)$ *such that*

$$\|S_N(\mathbf{x})\|_{1/p\text{-Hölder};[0,1]} \leq C_{14}\|\mathbf{x}\|_{1/p\text{-Hölder};[0,1]}.$$

4 Generalized Davie Estimates

In this section we show that the step-N Euler approximation is a good approximation to ODE solutions in small time, even if we control only the homogenous $1/p$-Hölder norm of $S_N(x)$. In the case of $N = 1,2$ this result is due to A. M. Davie, [7]. The existence of geodesics associated to the Carnot-Caratheodory metric is our main tool to generalize his results to the step-N case.

Recall that a control ODE driven by Lip^N vector fields has the step-N Euler approximation

$$\pi(s, y_s; x)_{s,t} \approx I\left[y_s, N, S_N(x)_{s,t}\right].$$

The Geodesic Existence theorem, applied to $\mathbf{g} = S_N(x)_{s,t}$, yields the shortest path in $\mathbb{R}^d$ whose iterated integrals mimick the first N iterated integrals of

the path segement $x|_{[s,t]}$. We called this path $x^{s,t} = (x_u^{s,t})_{u\in[s,t]}$. By construction, its step-N Euler approximation over $[s,t]$ is exactly equal to $I^{y_s,N,S_N(x)_{s,t}}$ and we are led to the equally good *step-N geodesic approximation*

$$\pi\left(s, y_s; x\right)_{s,t} \approx \pi\left(s, y_s; x^{s,t}\right)_{s,t}.$$

This step-N approximation is sometimes easier to handle. It also respect the geometry given by the vector fields. Below, we shall use both. As last preparation for the main result of this section, we need to understand the regularity $y \mapsto I[y, N, \mathbf{g}]$.

Lemma 15 *Assume that $(V_i)_{1\le i\le d} \in \mathrm{Lip}^N(\mathbb{R}^e)$. For an element $\mathbf{g} \in G^N(\mathbb{R}^d)$,*

$$\left|I^{y,N,\mathbf{g}} - I^{\tilde{y},N,\mathbf{g}}\right| \le C_{15}\left|y-\tilde{y}\right|\left(\|\mathbf{g}\| + \|\mathbf{g}\|^N\right)$$

where C_{15} depends on N and the Lip^N norm of the vector fields.

Proof. By definition of the Euler approximation $I^{y,N,\mathbf{g}}$

$$I^{y,N,\mathbf{g}} - I^{\tilde{y},N,\mathbf{g}} = \sum_{k=1}^{N} \sum_{\substack{i_1,\cdots,i_k\\ \in\{1,\cdots,d\}}} [V_{i_1}\cdots V_{i_k} H(y) - V_{i_1}\cdots V_{i_k} H(\tilde{y})]\mathbf{g}^{k,i_1,\cdots,i_k}.$$

Since $y \mapsto V_{i_1}\cdots V_{i_k} H(y)$ is Lipschitz,

$$\left|I^{y,N,\mathbf{g}} - I^{\tilde{y},N,\mathbf{g}}\right| \le C_{15}^1 \sum_{k=1}^{N} \sum_{\substack{i_1,\cdots,i_k\\ \in\{1,\cdots,d\}}} \left|y-\tilde{y}\right| \left|\mathbf{g}^{k,i_1,\cdots,i_k}\right|.$$

From equivalence of homogenous norms, $\left|\mathbf{g}^{k,i_1,\cdots,i_k}\right| \le C_{15}^2 \|\mathbf{g}\|^k$ and hence

$$\left|I^{y,N,\mathbf{g}} - I^{\tilde{y},N,\mathbf{g}}\right| \le C_{15}^3 \left|y-\tilde{y}\right| \max\left\{\|\mathbf{g}\|^N, \|\mathbf{g}\|\right\}.$$

■

The next lemma is technical but very important. It quantifies the quality of step-N Euler and geodesic approximations and gives ODE bounds which do not blow up with the Lipschitz norm of the driving signal. Recall that $\omega(s,t) \equiv t-s$ although the proof can adapted to general super additive control function [17].

Lemma 16 (Generalized Davie Lemma) *Let $p \ge 1$ and $(V_i)_{1\le i\le d} \in \mathrm{Lip}^N(\mathbb{R}^e)$ for some integer $N > p-1$. Assume that*
(i) $x : [0,1] \to \mathbb{R}^d$ is a Lipschitz path with step-N lift $\mathbf{x} = S_N(x)$ and $\|\mathbf{x}\|_{1/p\text{-Hölder}} \le M_1$.
(ii) $y_0 \in \mathbb{R}^e$ with $|y_0| \le M_2$; Then, there exists a positive constant $C_{16} = C_{16}(M_1)$, also dependent on p, N, M_2 and the vector fields $V_1, ..., V_d$ but ***not*** *dependent on the Lipschitz norm of x, such that for all $0 \le s \le t \le 1$*

$$\left|\pi\left(0, y_0; x\right)_{s,t}\right| < C_{16}(M_1)\,\omega(s,t)^{\frac{1}{p}}, \tag{2}$$

and with $\theta = \frac{N+1}{p} > 1$,

$$\left| \pi\left(0, y_0; x\right)_{s,t} - I^{y_s, N, \mathbf{x}_{s,t}} \right| \leq C_{16}\left(M_1\right) \omega\left(s,t\right)^{\theta}. \tag{3}$$

Moreover, if we assume
(iii) $x^{s,t} : [s,t] \to \mathbb{R}^d$ *are Lipschitz paths such that* $S_N\left(x^{s,t}\right)_{s,t} = \mathbf{x}_{s,t}$ *and such that*

$$\int_s^t \left| dx^{s,t} \right| \leq K M_1 \omega\left(s,t\right)^{1/p} \tag{4}$$

for some positive real K *then*

$$\left| \pi\left(0, y_0; x\right)_{s,t} - \pi\left(s, y_s; x^{s,t}\right)_{s,t} \right| \leq C'_{16}\left(M_1\right) \omega\left(s,t\right)^{\theta}, \tag{5}$$

with $C'_{16} = C'_{16}\left(M_1\right)$ *also dependent on* p, N, M_2, K *and the vector fields* $V_1, ..., V_d$.

Remark 17 *By hypothesis,* $\left\| \mathbf{x}_{s,t} \right\| \leq M_1 \omega\left(s,t\right)^{1/p}$ *and we can find a (projected) geodesic* $\gamma^* \colon [s,t] \to \mathbb{R}^d$ *with signature* $\mathbf{x}_{s,t}$ *and length* $\int_s^t \left| d\gamma^* \right| = \left\| \mathbf{x}_{s,t} \right\| \leq M_1 \omega\left(s,t\right)^{1/p}$. *In other words, paths* $\left\{ x^{s,t} \right\}$ *as postulated in (iii) always exist, even for* $K = 1$. *(By not fixing* K *we get a more general approximation result and highlight what is needed in the proof.)*

Proof. Without loss of generality, we assume that $M_1 \geq 1$ (otherwise set $M_1 = 1.$).

Write $y_t = \pi\left(0, y_0; x\right)_t$ and $\Gamma_{s,t} = y_{s,t} - \pi\left(s, y_s; x^{s,t}\right)_{s,t}$. We first show (5) and divide the argument in two steps.

First Step: Fix $0 \leq s < t < u \leq 1$. We try to control $\Gamma_{s,u}$ in terms of $\Gamma_{s,t}$ and $\Gamma_{t,u}$. To this end, it is useful to define $x^{s,t,u}$ to be the concatenation of $x^{s,t}$ and $x^{t,u}$. Observe that $x^{s,t,u}|_{[s,u]}$ has the step-N signature $\mathbf{x}_{s,t} \otimes \mathbf{x}_{t,u} = \mathbf{x}_{s,u}$ and

$$\int_s^u \left| dx^{s,t,u} \right| = \int_s^t \left| dx^{s,t} \right| + \int_t^u \left| dx^{t,u} \right| \leq 2 K M_1 \omega\left(s,u\right)^{1/p}. \tag{6}$$

By uniqueness of ODE solutions,

$$\pi\left(s, y_s; x^{s,t,u}\right)_r = \begin{cases} \pi\left(s, y_s; x^{s,t}\right)_r & \text{if } r \in [s,t] \\ \pi\left(t, \pi\left(s, y_s; x^{s,t}\right)_r; x^{t,u}\right)_r & \text{if } r \in [t,u]. \end{cases}$$

We have

$$\begin{aligned} -\Gamma_{s,u} + \Gamma_{s,t} + \Gamma_{t,u} &= \pi\left(s, y_s; x^{s,u}\right)_{s,u} - \pi\left(s, y_s; x^{s,t}\right)_{s,t} - \pi\left(t, y_t; x^{t,u}\right)_{t,u} \\ &= \pi\left(s, y_s; x^{s,u}\right)_{s,u} - \pi\left(s, y_s; x^{s,t,u}\right)_{s,u} \\ &\quad + \pi\left(s, y_s; x^{s,t,u}\right)_{s,u} - \pi\left(s, y_s; x^{s,t}\right)_{s,t} - \pi\left(t, y_t; x^{t,u}\right)_{t,u} \end{aligned}$$

By defintion of $\pi\left(s, y_s; x^{s,t,u}\right)$,

$$\pi\left(s, y_s; x^{s,t,u}\right)_{s,u} - \pi\left(s, y_s; x^{s,t}\right)_{s,t} = \pi\left(t, \pi\left(s, y_s; x^{s,t}\right)_t; x^{t,u}\right)_{t,u},$$

hence,

$$\begin{aligned}-\Gamma_{s,u}+\Gamma_{s,t}+\Gamma_{t,u} &= \pi\left(s,y_s;x^{s,u}\right)_{s,u}-\pi\left(s,y_s;x^{s,t,u}\right)_{s,u} \\ &\quad +\pi\left(t,\pi\left(s,y_s;x^{s,t}\right)_t;x^{t,u}\right)_{t,u}-\pi\left(t,y_t;x^{t,u}\right)_{t,u}.\end{aligned}$$

In particular, from lemma 5, we have

$$\begin{aligned}&\left|\pi\left(s,y_s;x^{s,u}\right)_{s,u}-\pi\left(s,y_s;x^{s,t,u}\right)_{s,u}\right| \\ &\qquad \leq \left|\pi\left(s,y_s;x^{s,u}\right)_{s,u}-I^{y_s,N,\mathbf{x}_{s,u}}\right|+\left|\pi\left(s,y_s;x^{s,t,u}\right)_{s,u}-I^{y_s,N,\mathbf{x}_{s,u}}\right| \\ &\qquad \leq 2C_{16}^{1}\left(M_1\right)\omega\left(s,u\right)^{\frac{N+1}{p}}=2C_{16}^{1}\left(M_1\right)\omega\left(s,u\right)^{\theta}.\end{aligned}$$

using (4) and (6). Then,

$$\begin{aligned}&\left|\pi\left(t,\pi\left(s,y_s;x^{s,t}\right)_t;x^{t,u}\right)_{t,u}-\pi\left(t,y_t;x^{t,u}\right)_{t,u}\right| \\ &\leq \left|\pi\left(t,\pi\left(s,y_s;x^{s,t}\right)_t;x^{t,u}\right)_{t,u}-I^{\pi\left(s,y_s;x^{s,t}\right)_t,N,\mathbf{x}_{t,u}}\right|+\left|\pi\left(t,y_t;x^{t,u}\right)_{t,u}-I^{y_t,N,\mathbf{x}_{t,u}}\right| \\ &\quad +\left|I^{\pi\left(s,y_s;x^{s,t}\right)_t,N,\mathbf{x}_{t,u}}-I^{y_t,N,\mathbf{x}_{t,u}}\right|.\end{aligned}$$

Once again, by lemma 5,

$$\begin{aligned}&\left|\pi\left(t,\pi\left(s,y_s;x^{s,t}\right)_t;x^{t,u}\right)_{t,u}-I^{\pi\left(s,y_s;x^{s,t}\right)_t,N,\mathbf{x}_{t,u}}\right|+\left|\pi\left(t,y_t;x^{t,u}\right)_{t,u}-I^{y_t,N,\mathbf{x}_{t,u}}\right| \\ &\qquad\qquad \leq 2C_{16}^{2}\left(M_1\right)\omega\left(t,u\right)^{\frac{N+1}{p}}\leq 2C_{16}^{2}\left(M_1\right)\omega\left(s,u\right)^{\theta}.\end{aligned}$$

Finally, by lemma 15,

$$\begin{aligned}\left|I^{\pi\left(s,y_s;x^{s,t}\right)_t,N,\mathbf{x}_{t,u}}-I^{y_t,N,\mathbf{x}_{t,u}}\right| &\leq C_{16}^{3}\left|\Gamma_{s,t}\right|\max\left(M_1\omega\left(t,u\right)^{1/p},M_1^N\omega\left(t,u\right)^{N/p}\right) \\ &\leq C_{16}^{3}\left|\Gamma_{s,t}\right|M_1^N\max\left(\omega\left(t,u\right)^{1/p},\omega\left(t,u\right)^{N/p}\right) \\ &\leq 2C_{16}^{3}\left(M_1\right)\left|\Gamma_{s,t}\right|\omega\left(t,u\right)^{1/p}\end{aligned}$$

using that $\omega\left(t,u\right)\leq 1$. Putting the pieces together, we have

$$\left|-\Gamma_{s,u}+\Gamma_{s,t}+\Gamma_{t,u}\right|\leq\left(2C_{16}^{1}\left(M_1\right)+2C_{16}^{2}\left(M_1\right)\right)\omega\left(s,u\right)^{\theta}+2C_{16}^{3}\left(M_1\right)\left|\Gamma_{s,t}\right|\omega\left(t,u\right)^{1/p}.$$

It follows that

$$\begin{aligned}\left|\Gamma_{s,u}\right| &\leq \left|-\Gamma_{s,u}+\Gamma_{s,t}+\Gamma_{t,u}\right|+\left|\Gamma_{s,t}\right|+\left|\Gamma_{t,u}\right| \\ &\leq \left|\Gamma_{s,t}\right|\left(1+C_{16}^{4}\left(M_1\right)\omega\left(t,u\right)^{1/p}\right)+\left|\Gamma_{t,u}\right|+C_{16}^{4}\left(M_1\right)\omega\left(s,u\right)^{\theta}\ (7)\end{aligned}$$

<u>*Second Step:*</u> For $0\leq s<t<u\leq 1$ inequality (7) can be rewritten as

$$\begin{aligned}\frac{\left|\Gamma_{s,u}\right|}{\omega\left(s,u\right)^{\theta}} &\leq \frac{\omega\left(s,t\right)^{\theta}}{\omega\left(s,u\right)^{\theta}}\frac{\left|\Gamma_{s,t}\right|}{\omega\left(s,t\right)^{\theta}}\left(1+C_{16}^{4}\left(M_1\right)\omega\left(t,u\right)^{1/p}\right) \\ &\quad +\frac{\omega\left(t,u\right)^{\theta}}{\omega\left(s,u\right)^{\theta}}\frac{\left|\Gamma_{t,u}\right|}{\omega\left(t,u\right)^{\theta}}+C_{16}^{4}\left(M_1\right).\end{aligned}$$

Define for $r \in (0,1]$,

$$\varrho(r) = \sup_{\substack{0 \le s < t \le 1 \\ \omega(s,t) \le r}} \frac{|\Gamma_{s,t}|}{\omega(s,t)^{\theta}}.$$

Note that $\rho(r) < \infty$. Indeed, this follows from

$$\begin{aligned}
|\Gamma_{s,t}| &= \left| y_{s,t} - \pi\left(s, y_s; x^{s,t}\right)_{s,t} \right| \\
&\le \left| y_{s,t} - I^{y_s, N, S_N(x)_{s,t}} \right| + \left| \pi\left(s, y_s; x^{s,t}\right)_{s,t} - I^{y_s, N, S_N(x)_{s,t}} \right| \\
&\le C_{16}^5 \left(\int_s^t |dx_r| \right)^{N+1} + C_{16}^5 \left(\int_s^t \left| dx_r^{s,t} \right| \right)^{N+1} \\
&\le C_{16}^5 |x|_{Lip}^{N+1} |t-s|^{N+1} + C_{16}^5 K M_1 \omega(s,t)^{\frac{N+1}{p}}.
\end{aligned}$$

The problem with this bound is that it blows up with $|x|_{Lip}$. The argument which follows shows that, in fact, $\rho(r)$ will *not* blow up with $|x|_{Lip}$. Pick arbitrary points $s < u$ such that $\omega(s,u) \le r$, amd set $t = (s+u)/2$ so that

$$\omega(s,t) = \omega(t,u) = \frac{1}{2}\omega(s,u).$$

We obtain from inequality (7) that

$$\begin{aligned}
\frac{|\Gamma_{s,u}|}{\omega(s,u)^{\theta}} &\le \left(\frac{1}{2}\right)^{\theta} \varrho(r/2) \left(1 + C_{16}^4 (M_1) r^{1/p}\right) \\
&\quad + \left(\frac{1}{2}\right)^{\theta} \varrho(r/2) + C_{16}^4 (M_1) \\
&\le 2^{1-\theta} \varrho(r/2) \underbrace{\left(1 + C_{16}^4 (M_1) r^{1/p}\right)}_{\equiv \gamma(r)} + C_{16}^4 (M_1)
\end{aligned}$$

and taking the supremum over all $s < u$ with $\omega(s,u) \le r$ gives

$$\varrho(r) \le 2^{1-\theta} \varrho(r/2) \gamma(r) + C_{16}^4.$$

After n iterated uses of the inequality for ρ we find

$$\begin{aligned}
\varrho(r) &\le \left(2^{1-\theta}\right)^{n+1} \prod_{k=0}^{n} \gamma\left(\frac{r}{2^k}\right) \varrho\left(\frac{r}{2^{n+1}}\right) \\
&\quad + C_{16}^4 (M_1) \left[\sum_{k=0}^{n} \left(\left(2^{1-\frac{N+1}{p}}\right)^k \prod_{j=0}^{k-1} \gamma\left(\frac{r}{2^j}\right) \right) \right].
\end{aligned}$$

Let $C_{16}^5 (r, M_1) := \sum_{k=0}^{\infty} \left(\left(2^{1-\frac{N+1}{p}}\right)^k \prod_{j=0}^{k-1} \gamma\left(\frac{r}{2^j}\right) \right)$. Note that $\prod_{k=0}^{n} \gamma\left(r/2^k\right)$ is increasing in n and since $\gamma(r) \le e^{C_{16}^4 (M_1) r^{1/p}}$, the supremum over r of the

infinite product $\prod_{k=0}^{\infty}\gamma\left(r/2^k\right)$ is finite, which implies that $\sup_{0\leq r\leq 1}C_{16}^5\left(r,M_1\right)$ is also finite.

Hence,

$$\varrho\left(r\right)\leq C_{16}^5\left(M\right)\left(2^{1-\theta}\right)^{n+1}\varrho\left(\frac{r}{2^{n+1}}\right)+\frac{C_{16}^4\left(M_1\right)C_{16}^5\left(r,M_1\right)}{1-2^{1-\theta}}$$

and sending $n\to\infty$ leaves us with (note $\theta>1$ here),

$$\varrho\left(r\right)\leq\frac{C_{16}^4\left(M_1\right)C_{16}^5\left(r,M_1\right)}{1-2^{1-\theta}}.$$

From the very definition of ρ with $r=1$ we obtain

$$\left|\Gamma_{s,t}\right|\leq C_{16}^6\left(M_1\right)\omega\left(s,t\right)^{\theta}.$$

Third Step: Using lemma 3 and (4),

$$\begin{aligned}\left|\pi\left(s,y_s;x^{s,t}\right)_{s,t}\right| &\leq C_{16}^7\int_s^t\left|dx^{s,t}\right|\\ &\leq C_{16}^7KM_1\omega\left(s,t\right)^{1/p}.\end{aligned}$$

Then, for all $s,t\in[0,1]$

$$\begin{aligned}\left|y_{s,t}\right| &\leq \left|y_{s,t}-\pi\left(s,y_s;x^{s,t}\right)_{s,t}\right|+\left|\pi\left(s,y_s;x^{s,t}\right)_{s,t}\right|\\ &\leq C_{16}^6\left(M_1\right)\omega\left(s,t\right)^{\theta}+C_{16}^7KM_1\omega\left(s,t\right)^{1/p}\\ &\leq C_{16}^6\left(M_1\right)\omega\left(s,t\right)^{1/p}+C_{16}^7KM_1\omega\left(s,t\right)^{1/p}\\ &\equiv C_{16}^8\left(M_1\right)\omega\left(s,t\right)^{1/p}.\end{aligned}$$

Although C_{16}^8 manifestly depends on K, we may specialize the construction using geodesics $\{x^{s,t}\}$for which $K=1$. With such paths, $C_{16}^8\left(M_1\right)$ of course would not depend on K. In particular, the Hölder norm on y does not depend on K.

Fourth Step: Finally, (3) is obtained from (5) via triangle inequality and lemma 5, taking into account (4). ■

Corollary 18 *There exists a constant C_{18}, which may depend on p,N,M_2 and the vector fields $V_1,...,V_d$ so that for all $M_1\geq 1$*

$$C_{16}\leq C_{18}\exp\left(12N^2\ln\left(M_1\right)^2\right),$$

This implies (the O-notation being understood as $M_1\to\infty$)

$$\ln C_{16}\left(M_1\right)=O\left(\left(\ln M_1\right)^2\right).$$

The same estimates holds for C_{16}', allowing for additional dependence on K.

Proof. Inspection of the first step in the proof of Davie's lemma shows that

$$C_{16}^{1}(M_1), C_{16}^{2}(M_1), C_{16}^{3}(M_1), C_{16}^{4}(M_1) \leq C_{18}^{1} M_1^{N+1}.$$

The only difficulty is to control

$$\begin{aligned} C_{16}^{5}(1, M_1) &= 1 + \sum_{k=1}^{\infty} \left(\left(2^{1-\frac{N+1}{p}} \right)^k \prod_{j=0}^{k-1} \gamma\left(\frac{1}{2^j} \right) \right) \\ &\leq \sum_{k=0}^{\infty} \left(\left(2^{1-\frac{N+1}{p}} \right)^k \prod_{j=0}^{k-1} \left(1 + C_{18}^{1} M_1^{N+1} 2^{-jp} \right) \right). \end{aligned}$$

To understand the dependence of the right hand side on M_1, we define the function for some fixed $a \in (0,1)$,

$$\begin{aligned} \Lambda(k,b) &= a^k \Pi_{j=0}^{k-1} \left(1 + b2^{-pj} \right) \\ \Gamma(n,b) &= \sum_{k=0}^{n} \Lambda(k,b). \end{aligned}$$

We need to understand the dependence of $\lim_{n\to\infty} \Gamma(n,b)$ on b. One could use a naive approach (the one used in the previous proof) to get

$$\begin{aligned} \Lambda(k,b) &\leq a^k \Pi_{j=0}^{\infty-1} \left(1 + b2^{-pj} \right) \\ &\leq a^k \Pi_{j=0}^{\infty-1} \exp\left(b2^{-pj} \right) \\ &\leq a^k \exp\left(\frac{b}{1-2^{-p}} \right), \end{aligned} \tag{8}$$

and hence,

$$\lim_{n\to\infty} \Gamma(n,b) \leq \frac{1}{1-a} \exp\left(\frac{b}{1-2^{-p}} \right).$$

Unfortunately, the right hand side in the last equation grows to fast in b for our purposes. To obtain a better estimate, we first observe that

$$\Lambda(k,0) = a^k \text{ and } \frac{\partial}{\partial b} \Lambda(k,b) = \left(\sum_{j=0}^{k-1} \frac{2^{-pj}}{1+b2^{-pj}} \right) \Lambda(k,b).$$

Then we note that $x \mapsto 2^{-px} / (1 + b2^{-px}) = b^{-1} \left(1 - (1 + b2^{-px})^{-1} \right)$ is de-

creasing in x so that

$$\begin{aligned}
\sum_{j=0}^{k-1} \frac{2^{-pj}}{1+b2^{-pj}} &\leq \frac{1}{1+b} + \sum_{j=1}^{\infty} \frac{2^{-pj}}{1+b2^{-pj}} \\
&\leq \frac{1}{1+b} + \int_0^{\infty} \frac{2^{-px}dx}{1+b2^{-px}} \\
&= \frac{1}{1+b} + \frac{\ln(1+b)}{b\ln(2^p)} \\
&\leq 3\frac{\ln b}{b}
\end{aligned}$$

for all $b \geq e$. We also note that

$$f(b) := \exp\left(\frac{3}{2}(\ln b)^2\right) \text{ solves } \frac{\partial}{\partial b} f(b) = 3\frac{\ln b}{b} f(b).$$

and any other solution to this ODE must be a multiple of f. By ODE comparison we see that, for $b \geq e$,

$$\frac{\Lambda(k,b)}{\Lambda(k,e)} \leq \frac{f(b)}{f(e)} \leq e^{\frac{3}{2}(\ln b)^2},$$

which implies that, using (8)

$$\begin{aligned}
\Lambda(k,b) &\leq \Lambda(k,e)\, e^{\frac{3}{2}(\ln b)^2} \\
&\leq a^k \exp\left(\frac{e}{1-2^{-p}}\right) e^{\frac{3}{2}(\ln b)^2}
\end{aligned}$$

After summing over all non-negative integers k we see that

$$\lim_{n\to\infty} \Gamma(n,b) \leq \frac{\exp\left(\frac{e}{1-2^{-p}}\right)}{1-a} e^{\frac{3}{2}(\ln b)^2}.$$

Hence, we have proved that

$$\begin{aligned}
C_{16}^5(1, M_1) &\leq \frac{\exp\left(\frac{e}{1-2^{-p}}\right)}{1-2^{1-\frac{N+1}{p}}} \exp\left(\frac{3}{2}\ln\left(C_{18}^1 M_1^{N+1}\right)^2\right) \\
&\leq C_{18}^2 \exp\left(3(N+1)^2 \ln(M_1)^2\right) \\
&\leq C_{18}^2 \exp\left(6N^2 \ln(M_1)^2\right).
\end{aligned}$$

This lead to

$$\begin{aligned}
C_{16}(M_1) &\leq C_{18}^3 M_1^{N+1} \exp\left(6N^2 \ln(M_1)^2\right) \\
&\leq C_{18}^3 \exp\left(12N^2 \ln(M_1)^2\right), \text{ for } M_1 \geq 3.
\end{aligned}$$

By increasing C_{18}^3 if needed we can assume that this estimate holds for all $M_1 \geq 1$. Clearly, the same estimate holds for $C_{16}'(M_1)$. ■

5 Euler Estimates for Rough Differential Equations (RDEs)

We consider controlled differential equations in the sense of T. Lyons. The driving signal is assumed to be a weak geometric p-rough path with Hölder control $\omega(s,t) = t - s$. Recall that this means $\mathbf{x} : [0,1] \to G^{[p]}(\mathbb{R}^d)$ is $1/p$-Hölder continuous w.r.t. Carnot-Caratheodory metric on $G^{[p]}(\mathbb{R}^d)$. Lyons' theory [15, 17, 16] then implies existence and uniquess of a solution to the differential equations driven by $\mathbf{x}$ along vector fields $V_1, \dots V_d \in Lip^{p+\epsilon}(\mathbb{R}^e)$ started at some point $y_0 \in \mathbb{R}^e$ at time 0. This *RDE solution* is also a (weak) geometric p-rough path, over $\mathbb{R}^e$ instead of $\mathbb{R}^d$, denoted by

$$\boldsymbol{\pi}(0, y_0, \mathbf{x}) \equiv \mathbf{y},$$

with the same modulus of continuity as $\mathbf{x}$. For our application it will be sufficient to consider the *pathlevel RDE solution* (obtained by projection)

$$\pi(0, y_0, \mathbf{x}) \equiv y : [0,1] \to \mathbb{R}^e.$$

Thus, y is an $\mathbb{R}^e$-valued $1/p$-Hölder continuous path in the usual sense.

Theorem 19 *Fix an integer $N \geq [p] + 1$ and Lip^N-vector fields $V_1, ..., V_d$ on $\mathbb{R}^e$. Let $\mathbf{x}$ be a weak geometric p-rough path with $\|\mathbf{x}\|_{p,\omega} \leq M$. Then there exists a unique pathlevel RDE solution $\pi(0, y_0, \mathbf{x}) \equiv y$. Moreover, (a) there exists constant $C_{19} = C_{19}(M)$, also dependent on p, N, y_0 and $V_1, ..., V_d$, such that*

$$|y|_{1/p\text{-}H\ddot{o}lder;[0,1]} \leq C_{19}$$

and (b) a constant $C'_{19} = C'_{19}(M)$ with similar dependencies such that for all $0 \leq s < t \leq 1$,

$$\left| y_{s,t} - I\left[y_s, N, S_N(\mathbf{x})_{s,t} \right] \right| \leq C'_{19} \omega(s,t)^\theta \quad \text{with } \theta = \frac{N+1}{p} > 1.$$

Finally, keeping all parameters but M fixed,

$$\ln C_{19}, \ln C'_{19} = O\left((\ln M)^2 \right) \quad \text{as } M \to \infty.$$

Proof. $\mathrm{Lip}^{[p]+1}$-regularity is more than enough to ensure existence and uniqueness of RDE solutions, see [15, 17, 16].
(a) From Theorem 12 we can find Lipschitz paths x^n such that

$$S_{[p]}(x^n) \to \mathbf{x}$$

uniformly on $[0,1]$, such that

$$\left\| S_{[p]}(x^n) \right\|_{1/p\text{-Hölder}} \leq 3M \equiv M_1$$

The Universal Limit Theorem implies a forteriori that

$$\pi\left(0,y_0;x^n\right)\to\pi\left(0,y_0;\mathbf{x}\right)$$

uniformly on $[0,1]$. On the other hand, Davie's lemma implies that

$$\sup_n\left|\pi\left(0,y_0;x^n\right)\right|_{1/p\text{-Hölder}}\leq C_{19}^1<\infty$$

where C_{19}^1 is the constant $C_{16}=C_{16}\left(M_1\right)$ from lemma 16. It follows that

$$\left|\pi\left(0,y_0;\mathbf{x}\right)\right|_{1/p\text{-Hölder}}\leq C_{19}^1<\infty.$$

From corollary 18,

$$\ln C_{19}^1=O\left(\left(\ln M_1\right)^2\right)=O\left(\left(\ln M\right)^2\right).$$

(b) By lemma 14, a weak geometric p-rough path $\mathbf{x}$ with $\|\mathbf{x}\|_{1/p\text{-Hölder}}\leq M$ lifts uniquely to a path $S_N\left(\mathbf{x}\right)\equiv\bar{\mathbf{x}}:[0,1]\to G^N\left(\mathbb{R}^d\right)$ such that

$$\|\bar{\mathbf{x}}\|_{1/p\text{-Hölder}}\leq C_{19}^2M$$

for some constant $C_{19}^2=C_{19}^2\left(p,N\right)$. As in part (a) we can find Lipschitz paths $\bar{x}^n$ such that

$$S_N\left(\bar{x}^n\right)\to\bar{\mathbf{x}}$$

uniformly on $[0,1]$, such that

$$\|S_N\left(\bar{x}^n\right)\|_{1/p\text{-Hölder}}\leq 3C_{19}^2M\equiv M_1$$

Note that, by projection, $S_{[p]}\left(\bar{x}^n\right)\to\mathbf{x}$ uniformly on $[0,1]$ with uniform homogenous $1/p$-Hölder bounds. As before, the Universal Limit Theorem implies that

$$\pi\left(0,y_0;\bar{x}^n\right)\to\pi\left(0,y_0;\mathbf{x}\right)\quad\text{uniformly on }[0,1]$$

while Davie's lemma implies the existence of C_{19}^3, de facto $C_{16}=C_{16}\left(M_1\right)$ from lemma 16, such that, uniformly over n, and for all $0\leq s<t\leq 1$,

$$\left|\pi\left(0,y_0;\bar{x}^n\right)_{s,t}-I\left[\pi\left(0,y_0;\bar{x}^n\right)_s,N,S_N\left(\bar{x}^n\right)_{s,t}\right]\right|\leq C_{19}^3\omega\left(s,t\right)^\theta$$

with $\theta=\left(N+1\right)/p$. By continuity of the map

$$\left(z,\mathbf{g}\right)\in\mathbb{R}^e\times G^N\left(\mathbb{R}^d\right)\mapsto I\left[z,N,\mathbf{g}\right]\in\mathbb{R}^e,$$

we can send $n\to\infty$ to obtain

$$\left|\pi\left(0,y_0;\mathbf{x}\right)_{s,t}-I\left[\pi\left(0,y_0;\mathbf{x}\right)_s,N,\mathbf{x}_{s,t}\right]\right|\leq C_{19}^3\omega\left(s,t\right)^\theta.$$

Finally, as above,

$$\ln C_{19}^3=O\left(\left(\ln M_1\right)^2\right)=O\left(\left(\ln M\right)^2\right).$$

■

6 Asymptotic Expansions for RDE Flows

We now consider RDEs driven by a random geometric p-rough path $\mathbf{x} = \mathbf{x}(\omega)$ defined on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We shall assume that the r.v. $\|\mathbf{x}\|_{1/p\text{-Hölder};[0,1]}$ has Gauss tails since this is the case for all examples we have in mind: Enhanced Brownian motion $\mathbf{B}$, see [9, 8], Enhanced Fractional Brownian Motion $\mathbf{B}^H$ and other Enhanced Gaussian processes [6, 10] and Enhanced Markov processes with uniformly elliptic generated in divergence form [13]. However, the proof of the following theorem will make clear that the method works whenever the real-valued r.v. $\|\mathbf{x}\|_{1/p\text{-Hölder};[0,1]}$ has some exponential tail decay.

Theorem 20 *Let $p \in (2,3)$, $N \geq [p]+1$ and consider the random RDE solution $\pi(0, y_0, \mathbf{x})$ driven by the random geometric p-rough path $\mathbf{x}$ (along fixed* Lip^N *vector fields $V_1, ..., V_d$), assuming that $\|\mathbf{x}\|_{1/p\text{-Hölder};[0,1]}$ has Gauss tails,*

$$\exists \alpha > 0 : \mathbb{E}\left[\exp\left(\alpha \, \|\mathbf{x}\|^2_{1/p\text{-Hölder};[0,1]}\right)\right] < \infty.$$

Then

$$|\pi(0, y_0, \mathbf{x})|_{1/p\text{-Hölder};[0,1]} \in L^q(\Omega) \text{ for all } q \in [1, \infty).$$

Moreover, the remainder of the step-N Euler approximation is bounded in probability. More precisely, there is a constant C_{20} dependent on α, p, N, y_0 and $V_1, ..., V_d$ such that for $R \geq 1$ and all $t \in (0,1]$,

$$\mathbb{P}\left(\sup_{0\leq s\leq t}\left|\pi(0, y_0, \mathbf{x})_{0,s} - I\left[y_0, N, S_N(\mathbf{x})_{0,s}\right]\right| > Rt^{\frac{N+1}{p}}\right) \leq C_{20}\exp\left(-e^{(\ln R)^{1/2}/C_{20}}\right).$$

In particular, the l.h.s. tends to zero uniformly over $t \in (0,1]$ as $R \to \infty$ and the convergence is faster than any power of $1/R$.

Proof. By assumption, $M_1 = \max\left\{\|\mathbf{x}\|_{1/p\text{-Hölder};[0,1]}, 1\right\}$ has Gauss tails and $\mathbb{E}\left[\exp\left(\alpha M_1^2\right)\right] < \infty$. From Theorem 19,

$$\sup_{0\leq s\leq t}\left|\pi(0, y_0, \mathbf{x})_{0,s} - Iy_0, N, S_N(\mathbf{x})_{0,s}\right| \leq C'_{19}(M_1)\, t^{(N+1)/p}.$$

where

$$C'_{19}(M_1) \leq C^1_{20} e^{\left(C^1_{20}\ln M_1\right)^2}.$$

Therefore,

$$
\begin{aligned}
& \mathbb{P}\left[\sup_{0\le s\le t}\left|\pi\left(0,y_0,\mathbf{x}\right)_{0,s}-I^{y_0,N,S_N(\mathbf{x})_{0,s}}\right|>Rt^{\frac{N+1}{p}}\right] \\
\le\ & \mathbb{P}\left[C_{20}^1 e^{\left(C_{20}^1 \ln M_1\right)^2}>R\right] \\
=\ & \mathbb{P}\left[M>\exp\left(\frac{\sqrt{\ln\left(R/C_{20}^1\right)}}{C_{20}^1}\right)\right] \\
\le\ & \mathbb{E}\left[\exp\left(\alpha M^2\right)\right]\exp\left[-\alpha\exp\left(\frac{2\sqrt{\ln\left(R/C_{20}^1\right)}}{C_{20}^1}\right)\right] \\
\le\ & C_{20}^2\exp\left(-e^{(\ln(R))^{1/2}/C_{20}^2}\right),
\end{aligned}
$$

where the last estimate is valid for every $R \ge 1$ by choosing C_{20}^2 sufficiently large. ■

Remark 21 *The same result holds if we replace* $I\left[y_0,N,S_N\left(\mathbf{x}\right)_{0,s}\right]$ *by* $\pi\left(0,y_0,x^{0,s}\right)_{0,s}$, *where* $x^{0,s}$ *is a geodesic associated to the element* $S_N\left(\mathbf{x}\right)_{0,s}$ *of* $G^N\left(\mathbb{R}^d\right)$.

Remark 22 *Even in the case of Enhanced Brownian motion,* $\mathbf{x}=\mathbf{B}$, *probability estimates of the unrestricted event*

$$\left\{\sup_{0\le s\le t}\left|\pi\left(0,y_0,\mathbf{x}\right)_{0,s}-I\left[y_0,N,S_N\left(\mathbf{x}\right)_{0,s}\right]\right|>Rt^{\frac{N+1}{p}}\right\}$$

valid for all $(t,R)\in(0,1]\times[1,\infty)$ *appear novel compared to the results given in [1, 2, 4].*

The perhaps strongest estimate that has been extracted from Azencott's work in this context (see [4, p 235]) is the following: in our notation (recall that $\pi\left(0,y_0,\mathbf{B}\right)$ solves a Stratonovich stochastic differential equation): $\exists a,c>0:$ $\forall R\ge 0:$

$$\overline{\lim_{t\to 0}}\,\mathbb{P}\left(\sup_{0\le s\le t}\left|\pi\left(0,y_0,\mathbf{B}\right)_{0,s}-I^{y_0,N,S_N(\mathbf{B})_{0,s}}\right|>Rt^{\frac{N+1}{2}}\right)\le ce^{-\frac{R^a}{c}}. \tag{9}$$

(Note that the exponent of t is$(N+1)/2$ in contrast to $(N+1)/p$ in Theorem 20). We now show how (9) can be deduced from our general results.

Proposition 23 *We keep all assumptions of the preceding theorem but drive the RDE with Enhanced Brownian motion* $\mathbf{x}=\mathbf{B}$. *Then (9) holds with* $a=2/\left(N+1\right)$ *and* $c=C_{23}$ *depending on* N,y_0 *and* $V_1,\dots,V_d$.

Proof. Choose $p=p\left(N\right)$ s.t.

$$2<p<2\frac{N+2}{N+1}\le 3$$

Then there exists $\alpha = \alpha(p) > 0$ s.t. $\mathbb{E}\left[\exp\left(\alpha M^2\right)\right] < \infty$ where $M = \max\left\{\|\mathbf{B}\|_{1/p\text{-Hölder};[0,1]}, 1\right\}$. We set $\bar{\mathbf{B}} \equiv S_{N+1}(\mathbf{B})$, well-defined by Proposition 14. Then

$$\begin{aligned}\left|\pi(0, y_0, \mathbf{B})_{0,s} - I^{y_0,N,S_N(\mathbf{B})_{0,s}}\right| &\leq \left|\pi(0, y_0, \mathbf{B})_{0,s} - I^{y_0,N+1,\bar{\mathbf{B}}_{0,s}}\right| \\ &\quad + \left|I^{y_0,N+1,\bar{\mathbf{B}}_{0,s}} - I^{y_0,N,S_N(\mathbf{x})_{0,s}}\right|.\end{aligned}$$

From Theorem 19 and Proposition 14,

$$\left|\pi(0, y_0, \mathbf{x})_{0,s} - I^{y_0,N+1,\bar{\mathbf{B}}_{0,s}}\right| \leq C_{23}^1 s^{(N+2)/p} e^{\left(C_{23}^1 \ln M\right)^2}.$$

On the other hand

$$\begin{aligned}\left|I^{y_0,N+1,S_{N+1}(\mathbf{B})_{0,s}} - I^{y_0,N,S_N(\mathbf{x})_{0,s}}\right| &= \Big|\sum_{\substack{i_1,\ldots,i_{N+1}\\ \in\{1,\ldots,d\}}} V_{i_1}\cdots V_{i_{N+1}} H(y_0)\, \bar{\mathbf{B}}_{0,s}^{N+1,i_1,\cdots,i_{N+1}}\Big| \\ &\leq C_{23}^2 \sum_{i_1,\ldots,i_{N+1}\in\{1,\ldots,d\}} \left|\bar{\mathbf{B}}_{0,s}^{N+1,i_1,\cdots,i_{N+1}}\right| \\ &\leq C_{23}^3 \left\|\bar{\mathbf{B}}_{0,s}\right\|^{N+1}.\end{aligned}$$

Trivially, $\sup_{0\leq s\leq t} s^{(N+2)/p} = t^{(N+2)/p}$ and we are led to

$$\begin{aligned}&\mathbb{P}\left(\sup_{s\in[0,t]}\left|\pi(0, y_0, \mathbf{x})_{0,s} - I^{y_0,N,S_N(\mathbf{x})_{0,s}}\right| \geq R t^{\frac{N+1}{2}}\right) \\ \leq\ & \mathbb{P}\left(t^{\frac{N+2}{p}} C_{23}^1 e^{\left(C_{23}^1 \ln M\right)^2} + C_{23}^3 \sup_{s\in[0,t]}\left\|\bar{\mathbf{B}}_{0,s}\right\|^{N+1} \geq R t^{\frac{N+1}{2}}\right) \\ \leq\ & (1) + (2)\end{aligned}$$

where with $\nu = (N+2)/p - (N+1)/2 > 0$,

$$(1) = \mathbb{P}\left(t^{\nu} C_{23}^1 e^{\left(C_{23}^1 \ln M\right)^2} \geq \frac{R}{2}\right)$$

and

$$(2) = \mathbb{P}\left(C_{23}^3\left(t^{-1/2}\sup_{s\in[0,t]}\left\|\bar{\mathbf{B}}_{0,s}\right\|\right)^{N+1} \geq \frac{R}{2}\right).$$

Gauss tails of M are more than enough to asset that (1) tends to zero as $t \to 0$. As for (2), Brownian scaling shows that (2) is in fact independent of t and hence equal to

$$\mathbb{P}\left(C_{23}^3\left(\sup_{s\in[0,1]}\left\|\bar{\mathbf{B}}_{0,s}\right\|\right)^{N+1} \geq \frac{R}{2}\right) \leq \mathbb{P}\left(C_{23}^4 M^{N+1} \geq \frac{R}{2}\right)$$

since $\sup_{s\in[0,1]}\left\|\bar{\mathbf{B}}_{0,s}\right\| \leq \left\|\bar{\mathbf{B}}\right\|_{1/p\text{-Hölder};[0,1]} \leq C_{23}^5 \left\|\mathbf{B}\right\|_{1/p\text{-Hölder};[0,1]} \leq C_{23}^5 M$. Gauss tails of M now easly the claimed tail decay. ■

Remark 24 *Estimate (9) remains valid when* $I\left[y_0, N, S_N(\mathbf{B})_{0,s}\right]$ *is replaced by* $\pi\left(0, y_0; x^{0,s}\right)$ *where* $x^{0,s}$ *is a geodesic associated to* $S_N(\mathbf{x})_{0,s}$.

Remark 25 *Note that* $a = 2/(N+1)$ *is what one expects from integrability of the* $(N+1)^{th}$ *multiple Wiener-Itô integral which dominates the remainder as* $t \to 0$. *Theorem 20 gives an estimate valid uniformly in* $t \in (0,1]$ *and we would not expect an exponential tail. In fact, the given estimate implies a tail decay which is about as good as one can hope in absence of exponential decay and there seems little room for improvement.*

Remark 26 *Only in absence of a drift vector field* V_0 *does our step-N Euler approximation* $I\left[y_0, N, S_N(\mathbf{x})_{0,s}\right]$ *coincide precisely with the approximation of [1, prop 4.3] and [4, p 235]. When* $V_0 \neq 0$, *the methodology and results of* (p,q)*-rough paths [14] could be used to produce the same approximations as those in the above cited references. If one accepts a few additional terms in the step-N approximation, it may be simplest to deal with* $V_0 \neq 0$ *via an RDE driven by the canonically defined time-space rough path.*

Remark 27 *Proposition 23 is readily adapted to EFBM* $\mathbf{B}^H$ *with* $H > 1/4$ *and gives*

$$\overline{\lim_{t\to 0}}\mathbb{P}\left(\sup_{0\leq s\leq t}\left|\pi\left(0, y_0, \mathbf{B}^H\right)_{0,s} - I^{y_0, N, S_N\left(\mathbf{B}^H\right)_{0,s}}\right| > Rt^{H(N+1)}\right) \leq ce^{-\frac{R^a}{c}}.$$

Remark 28 *In comparison to the Taylor expansion of Azencott, the approximations by Ben Arous and Castell respect the geometry of the problem. This is also the case for our geodesic approximations although the efficient approximations of sub-Riemannian geodesics remains a numerical challenge.*

7 L^q Convergence in the Universal Limit Theorem

We now give a criterion for L^q-convergence in the Universal Limit Theorem.

Proposition 29 *Assume that for a random sequence of rough path*

$$\sup_n \left\|\mathbf{x}_n\right\|_{1/p\text{-Hölder};[0,1]} \equiv M$$

has a tail Gaus tail. Assume that $d_{1/p\text{-Hölder}}\left(\mathbf{x}_n, \mathbf{x}\right) \to 0$ *in probabililty. Then*

$$\left|\pi\left(0, y_0, \mathbf{x}_n\right) - \pi\left(0, y_0, \mathbf{x}\right)\right|_{1/p\text{-Hölder};[0,1]} \to 0 \text{ in } L^q\ \forall q \in [1,\infty).$$

Proof. By the universal limit theorem,

$$Z_n \equiv |\pi(0,y_0,\mathbf{x}_n) - \pi(0,y_0,\mathbf{x})|_{1/p\text{-Hölder};[0,1]} \to 0 \text{ in probability.}$$

At least along a subsequence, $Z_{n_k} \to 0$ a.s. as $k \to \infty$ and

$$|\pi(0,y_0,\mathbf{x})|_{1/p\text{-Hölder};[0,1]} = \lim_{k\to\infty} |\pi(0,y_0,\mathbf{x}_{n_k})|_{1/p\text{-Hölder};[0,1]}$$

and therefore

$$Z_n \le 2\sup_n |\pi(0,y_0,\mathbf{x}_n)|_{1/p\text{-Hölder};[0,1]} \le 2C$$

where $C = C(M) \in L^q\ \forall q \in [1,\infty)$ using Theorem 20. To obtain L^q convergence we have to check that $\{Z_n^q : n \ge 1\}$ is uniformly integrable. But this follows immediately from L^r-boundedness, $r > 1$, indeed

$$\mathbb{E}(|Z_n^q|^r) \le 2^{qr}\mathbb{E}(C(M)^{qr}) < \infty.$$

■

Example 30 *Let $\mathbf{x} = \mathbf{B}$ be Enhanced Brownian motion over $\mathbb{R}^d$ with*

$$\|\mathbf{x}\|_{1/p\text{-}Hölder;[0,1]} < \infty \text{ where } p \in (2,3).$$

Let (D_n) be a nested family of dissections of $[0,1]$ with mesh $|D_n| \to 0$ and define the lifted piecewise linear approximation

$$\mathbf{x}_n = S_2(\mathbb{E}[B|\sigma(B_t : t \in D_n)]).$$

Then observe that the area process of $\mathbf{x}_n$ is obtained by the Lévy area process condtioned on $\sigma(B_t : t \in D_n)$. Then

$$\sup_n \|\mathbf{x}_n\|_{1/p\text{-}Hölder;[0,1]} \le \sup_n \mathbb{E}\left[\|\mathbf{x}\|_{1/p\text{-}Hölder;[0,1]} |\sigma(B_t : t \in D_n)\right] \equiv \sup_n M(n) \equiv M < \infty.$$

Note that

$$\begin{aligned} M(n) &\equiv \mathbb{E}\left[\|\mathbf{x}\|_{1/p\text{-}Hölder;[0,1]} |\sigma(B_t : t \in D_n)\right] \\ &\le \mathbb{E}\left[\|\mathbf{x}\|^q_{1/p\text{-}Hölder;[0,1]} |\sigma(B_t : t \in D_n)\right]^{1/q} \end{aligned}$$

and from Doob's maximal inequality,

$$\begin{aligned} \mathbb{E}\left|\sup_n \|\mathbf{x}_n\|^q_{1/p\text{-}Hölder;[0,1]}\right| &\le \left(\frac{q}{q-1}\right)^q \sup_n \mathbb{E}(M(n)^q) \\ &\le \left(\frac{q}{q-1}\right)^q \mathbb{E}\left[\|\mathbf{x}\|^q_{1/p\text{-}Hölder;[0,1]}\right]. \end{aligned}$$

Noting that $(q/(q-1))^q$ stays bounded as $q \to \infty$ (in fact, it converges to e), we conclude that Gauss tails of M_1 imply Gauss tails of M. Finally, fix $p \in (2,3)$ and repeat the above argument for $\tilde{p} \in (2,p)$ replacing p. By interpolation

$$d_{1/p\text{-}Hölder}(\mathbf{x}_n,\mathbf{x}) \to 0 \text{ a.s.}$$

and uniform Gauss tails follow from the trivial estimate $\|\mathbf{x}_n\|_{1/p\text{-}Hölder;[0,1]} \le \|\mathbf{x}_n\|_{1/\tilde{p}\text{-}Hölder;[0,1]}$.

Remark 31 *The same argument works for other Enhanced Gaussian processes with martingale approximations such as fractional Brownian motion, see [10] for the case $H > 1/3$.*

References

[1] Azencott, R.: Formule de Taylor stochastique et développement asymptotique d'intégrales de Feynman. Séminaire de probabilités XVI, Springer Lect. Notes Math. 921, 237-285 (1982).

[2] Ben Arous, G.: Flots et Séries de Taylor Stochastique. Probab. Theory Relat. Fields 81, 29-77 (1989).

[3] Baudoin, F.: An Introduction to the Geometry of Stochastic Flows, Imperial College Press (2005).

[4] Castell, F.: Asymptotic expansions of stochastic flows. Probab. Theory Relat. Fields 96, 225-239 (1993).

[5] Coutin, L.; Friz, P.; Victoir, N.: Good Rough Path Sequences and Applications to Anticipating & Fractional Stochastic Calculus, ArXiv-preprint (2005)

[6] Coutin, L.; Qian Z.: Stochastic analysis, rough path analysis and fractional Brownian motions. Probab. Theory Relat. Fields 122, 108–140 (2002).

[7] Davie, A. M.: Differential equations driven by rough signals: an approach via discrete approximation. Preprint (2003).

[8] Friz, P., T. Lyons, D. Stroock: Levy's area under conditioning, Annales de l'Institut Henri Poincare (B) Probability and Statistics, Volume 42, Issue 1, 89-101 (2006).

[9] Friz, P., Victoir, N: Approximations of the Brownian rough path with applications to stochastic analysis, Annales de l'Institut Henri Poincare (B) Probability and Statistics, Volume 41, Issue 4, 703-724 (2005).

[10] Friz, P. Victoir, N: On the notion of Geometric Rough Paths. To appear in Probab. Theory Relat. Fields (2006).

[11] Goodman, R.: Filtrations and Asymptotic Automorphisms on Nilpotent Lie Groups, J.Diff.Geometry 12 (1977) 183-196.

[12] Kunita, H.: Stochastic Flows and Stochastic Differential Equations, Cambridge University Press (1992)

[13] Lejay, A.: Stochastic Differential Equations Driven by Processes Generated by Divergence Form Operators, preprint (2003).

[14] Lejay, A. Victoir, N. On (p,q)-rough paths. To appear in Journal of Differential Equations.

[15] Lyons, Terry J.: Differential equations driven by rough signals. Rev. Mat. Iberoamericana 14, no. 2, 215–310 (1998).

[16] Lyons, Terry J.: St. Flour lecture notes (2004).

[17] Lyons, T.; Qian, Z.: System Control and Rough Paths, Oxford University Press (2002).

[18] Montgomery, R.: A Tour of Subriemannian Geometry, Their Geodesics and Applications. Mathematical Surveys and Monographs, vol. 91, 2002

[19] Platen, E.: A Taylor formula for semimartingales solving a stochastic equation, 3rd conference on stochastic differential systems, Visegrad/Hongrie, 65-68, 1980.